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A Topological Framework for Approximation Spaces Induced by Finite Group Conjugacy

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ABSTRACT

This paper introduces a novel theoretical framework for constructing topological approximation spaces by leveraging the algebraic structure of finite groups. The core innovation lies in utilizing the conjugacy relation ($\star R$) as a fundamental equivalence relation to partition the universal set U. These conjugacy classes serve as the base for generating a specific topology ($\star\tau$) on the group. Within this hybrid environment, a new type of approximation is established based on topological interior and closure operators. The study demonstrates significant results, proving that every base element in this topology is a clopen set. Moreover, it has been shown that the induced topology τ_{conj} *yield the discrete topology if and only if underling groub is abelian, whereas non-abelian groups generate non discrete spaces with non-trivial rough boundaries. The analysis is extended to define a topological rough memebership function* ($\mu_X^{\tau_{conj}}$) that bridges algebraic centralizer indices with topological neighborhood structures to quantify uncertainty. *To validate the efficiency and applicability of this framework, comprehensive analytical examples are provided using the Dihedral group (D_3) and the Quaternion group (Q_8).*

1. Introduction

Since the foundational work of the early 20th century, topology has become a cornerstone of modern mathematics, providing indispensable tools to study continuity, connectedness, and spatial relationships [2, 11]. On a parallel trajectory, Rough Set Theory, introduced by Pawlak in 1982, offers a robust mathematical approach for handling vague, uncertain, and incomplete information systems [3, 6]. The core concept relies on approximating an imprecise target set using lower and upper approximations defined via an equivalence relation (indiscernibility relation) on a universe U, thereby forming an approximation space (U,R) [3,12]. In classical rough set models, indiscernibility relations are typically constructed from attribute-value tables or arbitrary binary relations. However, incorporating structured algebraic systems—such as Finite Group Theory—into approximation spaces provides additional mathematical symmetries and algebraic properties. Conjugacy relations in finite groups inherently partition the universe into well-defined equivalence classes with rich structural properties [1,13].

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Research Gap and Motivation

Although many studies have explored extending rough sets through topological ideas [5, 9, 16], most of these models lean heavily on pure topology without really tapping into the underlying algebraic structures. To the best of the author's knowledge, the systematic utilization of group conjugacy relations to construct topological approximation spaces and investigate their hybrid algebraic-topological structures has received limited attention in the literature. Addressing this gap establishes a unified framework where fundamental algebraic invariants directly govern topological and rough set approximations.

Main Contributions The main goals and key contributions of this study are:

1- Building a group-induced topological approximation space (U,⋆R ,⋆τ) in which conjugacy classes are used as a base to build the topology ⋆τ and group conjugacy defines the equivalency relation ⋆R.
2- rigorously demonstrating the essential characteristics of the induced topology, such as the collapse to a discrete topology in the case of Abelian groups and the clopen property of base elements
3- A new topological rough membership function $\mu_X^{\star\tau}$ is defined, and its clear properties in capturing set boundary uncertainty are shown.
4- Giving examples based on D_3 and Q_8 to show how upper/lower approximations and membership values can be computed practically

This paper is tight as tails:

- This paper's remaining sections are organized as follows: Basic introductions to group theory, topology, and rough sets are given in Section 2. The group-induced approximation space is formalized in Section 3. The topological framework and an analysis of exact and rough sets are presented in Section 4. The topological rough membership function is described in depth in Section 5. A thorough examination of comparative analysis and its limits is given in Section 6, and the study is concluded in Section

2. Preliminaries

2.1 Fundamental of Group Theory

Definition 2.1.1 [1,13]: A group $(G,\star)$ is a set G closed under a binary opration $\star$ sufficient: closure, associativity, identity element existence for all elements. A finite group has a finite set G.

Definition 2.1.2 [1]: A group $(G,\star)$ is abelian (or commutate*i*ve) if $a \star b. = b \star a.\ \ \forall a, b \in G$

Definition 2.1.3 [1]: The Center of a Group G, denoted by $C(G)$, is the set of elements that commute with each component of

$$G: C\ (G) = \{a \in G. : a \star x = x\ \star a\ \ \forall x\ \in G\}.$$

Example 2.1.4:

Let's take the $Klein\ Four - Group(V_4)$, which is the symmetry set of a rectangle (that is not a square).

The group consists of 4 symmetry transformations of the rectangle:

- $\boldsymbol{e}$: The identity transformation (do nothing).
- $\boldsymbol{a}$: Reflection across the horizontal axis.
- $\boldsymbol{b}$: Reflection across the vertical axis.
- $\boldsymbol{c}$: Rotation by 180° about the center.

1. Closure Under the Binary Operation

Definition: When we apply the group's binary operation (here: composition of transformations ∘) to any two elements in the group, the result must be another element that is already in the group.

Group Operation Table (Cayley Table) for V_4:

∘	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Opinion: We can see that the result of composing any two transformations (e.g., $a \circ b = c$) is always one of the 4 elements $\{e, a, b, c\}$. No result falls outside the group. Therefore, the group is closed under the binary operation.

2. Commutatively

Definition: For group is commutative, then $x \circ y. = .y \circ x$ for any two elements x and y in the group.

3. Center of a Group

Definition: In center of a gr*o*up $C(G)$ its' a set of all elements that exchange with each other part in the group. That is,

$$C(G) = \{x \in G: x \circ g = g \circ x, \forall g \in G\}.$$

Since, every element in V_4 commutes with every other element, the center of this group is the entire group: $C(V_4) = V_4$.

Definition 2. 1. 5 [1,13]: The D*i*hedral Gr*o*up $D.n$ is the group of asymmetries of a ordered n-sided polygon, consisting of n rotations and n reflections, with order $2n$.

Example 2.1. 6 [1]:

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The dihedral group $D3$ of order 6 can be represented as $D_3 = \{e, r, R, s, sr, sR\}$, where r is a rotation and s is a reflection.

Definition 2.1. 7 [1]: The Quaternion Group Q_8 is a non-Abelian group of order 8: $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication rules $i^2 = j^2 = k^2 = ijk = -1$

2.2 Fundamentals of Topological Spaces

Definition 2.2.1 [2,11]: A Topological Universe be a duo (U, τ) where U be a set and τ is a collection of subsets of U (called open sets) satisfying: U and $\emptyset$ are in τ ; arbitrary mergers of open sets are open; finite intersections of open sets are open. A set is closed if its complement is open.

Definition 2.2.2 [2,11]: The Interior of a set $X \subseteq U$, denoted $X \circ$ .then a unification of altogether open sets contained in X. The closure of X, denoted X, is the intersection of all closed sets containing X.

Definition 2.2.3 [11]: A base B for a topology τ is a collection of open sets such that every open set in τ is a union of elements of B.

Definition 2.2.4 [15]: An equivalence relation R on U is binary relation that be $reflexive, symmetric, and\ transitive$. The equality class for element $a \in U\ is\ [a]R = \{x \in U: (x, a) \in R\}$. A set of very similarity classes, U/R, forms a partition of U.

Example 2.2.5 (Detailed Illustration):

Let $U = \{1,2,3\}$ and $\tau = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, U\}$

- **Step 1** (Topology Verification): $\boldsymbol{U \in \tau\ and\ \emptyset \in \tau}$. Unions such as : $\{1\} \cup \{2\} = \{1\ 2\} \in \tau$ and intersections $\{\mathbf{1}, \mathbf{2}\} \cap \{\mathbf{1}, \mathbf{3}\} = \{\mathbf{1}\} \in \boldsymbol{\tau}$ **hold.**
- **Step 2 (Interior Calculation):** For X = {1, 2}, open subsets contained in X are: $\emptyset, \{1\}, \{2\}, \{1, 2\}$ thus $X^{\circ} = \{1\} \cup \{2\} \cup \{1.2\} = \{1,2\}$. Since, $X = X^{\circ}, X$ is an open set in τ.
- **Step 3 (Closure Calculation):** Closed sets are complements of open sets: {2}, {3}, {1, 3}, {2, 3}, ∅. Closed sets that contain $X = \{1,2\}$ are only U. Therefore $\overline{\boldsymbol{X}} = \boldsymbol{U}$
- **Step 4 (Basis Construction):** A base for τ is $\beta = \{\{1\}, \{2\}, \{1,3\}\}$, as every member of τ is a union of elements in β.

1.3 Fundamentals of Rough Set Theory

Definition 2.3.1 [3,6]: If R be an equivalence relative on U. The pair (U, R) is an approximation space.. Aimed at every $X \subseteq U$:

- Upper Approximation: $\overline{R}(X) = \{x \in U: [x]R \cap X \neq \emptyset\}$
- Lower Approximation: $\underline{R}(X.) = \{x \in .U: [.x]R \subseteq X\}$
- Boundary Region: $B\ N_R(X) = \overline{R}(X) - \underline{R}(X)$
- Positive Region: $POS(X) = \underline{R}(X)$
- Negative Region: $NEG(X) = U - \overline{R}(X)$
- Accuracy Measure: $\alpha R(X) = |\underline{R}(X)| / |\overline{R}(X)|$ where $0 \leq \alpha R(X) \leq 1$.

If $\underline{R}(X) \neq \overline{R}(X)$, then X is rough set; otherwise, it is an exact set [12].

Example 2.3.2 (Approximation Space & Core Concepts)

- If, $U = \{1, 2, 3, 4, 5\}$ (The Universal / Universal Set),
- R its' similarity relative that divider U for the following equivalence classes.
- $\{1\} = \{1, 3, 5\}$
- $\{2\} = \{2, 4\}$

Hence, hold approximation space is (U, R).

If income the set $X = \{1, 2, 4\}$ and calculate its approximations:

1. **Lower Approximation** $(\underline{\boldsymbol{R}}(\boldsymbol{X}) = \{\boldsymbol{x} \in \boldsymbol{U}: [\boldsymbol{x}]\boldsymbol{R} \subseteq \boldsymbol{X}\})$
   - For element 1: $[1] = \{1, 3, 5\}$ is not contained in X (because 3,5 are not in X) $\leftarrow 1$ is not included
   - For element 2: $[2] = \{2, 4\}$ is not contained in $X \leftarrow 2$ is included
   - For element 3: $[3] = \{1, 3, 5\}$ is not contained in $X \leftarrow 3$ is not included
   - For element 4: $[4] = \{2, 4\}$ is contained in $X \leftarrow 4$ is included
   - For element 5: $[5] = \{1, 3, 5\}$ is not contained in $X \leftarrow 5$ is not included

   $$\underline{\boldsymbol{R}}(\boldsymbol{X}) = \{\mathbf{2}, \mathbf{4}\}$$
2. **Upper Approximation** $(\overline{\boldsymbol{R}}(\boldsymbol{X}) = \{\boldsymbol{x} \in \boldsymbol{U}: [\boldsymbol{x}]\boldsymbol{R} \cap \boldsymbol{X} \neq \emptyset\})$
   - For element 1 : $[1] \cap X = \{1,3,5\} \cap \{1,2,4\} = \{1\} \neq \emptyset \leftarrow 1$ is included
   - For element 2 : $[2] \cap X = \{2, 4\} \cap \{1,2,4\} = \{2, 4\} \neq \emptyset \leftarrow 2$ is included
   - For element 3 : $[3] \cap X = \{1,3,5\} \cap \{1,2,4\} = \{1\} \neq \emptyset \leftarrow 3$ is included
   - For element 4 : $[4] \cap X = \{2, 4\} \cap \{1,2,4\} = \{2,4\} \neq \emptyset \leftarrow 4$ is included
   - For element 5 : $[5] \cap X = \{1,3,5\} \cap \{1,2,4\} = \{1\} \neq \emptyset \leftarrow 5$ is included

   $$\overline{\boldsymbol{R}}(\boldsymbol{X}) = \boldsymbol{U} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$$
3. **Boundary Region**

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$$B\ N_R(X.) = \overline{R}(X) - .\underline{R}(X)$$
$$= \{.1,2,3,4,5\} - \{2,4\}$$
$$= \{1,3,5\}$$

- Positive Region: $POS(X) = \underline{R}(X) = \{2,4\}$
- Negative Region: $NEG(X) = U - \overline{R}(X) = \emptyset$

4. Accuracy Measure

$\alpha R(X) = \left|\underline{R}(X)\right| / \left|\overline{R}(X)\right|$ = 2/5 = 0.4

Definition 2. 3.3 [10]: the Rough Membership Function for a set $X \subseteq U$ is defined as:

$$\mu_X^R : U \longrightarrow [0,1], \qquad \mu_{X.}^R(x) = \frac{|[x.]_R \cap X|}{|.[x]_R|}$$

It satisfies: $\mu_X^R = 1 \Longrightarrow x \in \overline{R}(X)$; $\mu_X^R = 0 \Longrightarrow x \notin \overline{R}(X)$.

Example 2.3.4 (Rough Set vs. exact Set): Let's use the same approximation space $(U, .R)$ as before then: $U = \{1,2,3,4,5\}$

Equivalence classes: $[1] = \{1,3,5\}$

$[2] = \{2,4\}$

Case A: Rough Set

Let's take $X = \{1,2,4\}$ (same as previous example)

- $\underline{R}(X) = \{2,4\}$
- $\overline{R}(X) = \{1,2,3,4,5\}$. Since $\underline{R}(X.) \neq .\overline{R}(X)$. Then X its' $Rough\ Set$.

Case B: Exact Set

Let's take $Y = \{1,3,5\}$. Now let's calculate the approximations of Y:

- $\underline{R}(Y) = \{x \in U : [x]R \subseteq X\}$
- $[1] = \{1,3,5\} \subseteq Y \leftarrow 1$ is included
- $[2] = \{2,4\}$ is not contained in $Y \leftarrow 2,4$ are not included
- $[3] = \{1,3,5\} \subseteq Y \leftarrow 3$ is included
- $[4] = \{2,4\}$is not contained in $Y \leftarrow 2,4$ are not included
- $[5] = \{1,3,5\} \subseteq Y \leftarrow 5$ is included
- $\underline{R}(.X) = \{1,3,5\}$

  $.\overline{\boldsymbol{R}}(\boldsymbol{X}) = .\{\boldsymbol{x} \in \boldsymbol{U} : [\boldsymbol{x}]\boldsymbol{R} \cap \boldsymbol{Y}. \neq \emptyset\}$
- $[1] \cap Y = \{1,3,5\} \neq \emptyset \leftarrow 1$ is included
- $[2] \cap Y = \{1,3,5\} \cap \{2,4\} = \emptyset \leftarrow 2,4$ are not included
- $[3] \cap Y = \{1,3,5\} \neq \emptyset \leftarrow 3$ is included
- $[4] \cap Y = \emptyset \leftarrow 4$ is not included
- $[5] \cap Y = \{1,3,5\} \neq \emptyset \leftarrow 5$ is included
- $\overline{R}(X) = \{1,3,5\}$. Since $\underline{R}(X) = \overline{R}(X) = \{1,3,5\}$. Then Y is an Exact Set.

3. Approximation Space Induced by Finite Group

Definition 3.1 (Group-Induced Approximation Space):

Let $(U, \star)$ be a finite group. The conjugacy relation $\star$ $R \subseteq .U \times U$ is definite as:

$\star R = \{.(u_1, u_2) \in .U \times U : \exists g \in U$ such that $u_1 = g \star u_2 \star 1/g\}$

Then, duo $(U, \star R)$ its' named an approximation space induced by the finite group.

Theorem 3.2:

The conjugacy relation $\star R$ is an equivalence relation on U.

Proof:

- Reflexive: For any $a \in U, a = e \star a \star \frac{1}{e}$ (where e is identity).
  So, $(.a, a) \in \star R$.
- Symmetric: If $(a, b) \in \star R$, then $a = gb1/g$ for some g.then
  $b = g^{-1}a\ (g^{-1})^{-1} = g^{-1}a\ g, so\ (b, a) \in \star R$.
- Transitive: If $(a, b) \in \star R$ and If $(b, c) \in \star R$, then $a = gb\ g^{-1}\ and$
  $b = hc\ h^{-1}$ substituting, $a = g(hc\ h^{-1})g^{-1} = (gh)c(h^{-1}g^{-1}) = (gh)c(gh)^{-1}.thus, (a, c) \in \star Rwith\ k = gh$

Example 3.3: When $U=D_3 = \{e, r, r^2, s, sr, sr^2\}$ The conjugacy classes are:

- $[e_{\star R}] = \{e\}$
- $[r_{\star R}] = [r^2{}_{\star R}] = \{r, r^2\}$
- $[s_{\star R}] = [rs_{\star R}] = \{s, sr, sr^2\}$

These classes form the equivalence relation $\star R$, making $(D_3, \star R)$ an approximation space.

4. Topological Approximation Space Via Finite Groups

We now construct a topology form the group-induce approximation space.

Definition 4.1 (Topological Approximation Space): If $(U, \star)$ its' a finite group and $(U, \star R)$ its induced approximation space. And clan of equivalence classes $U/\star R = \{[x]_{\star R} : x \in U\}\}$ forms a base for a topology $\star \tau$ on U. The triple $(U, \star R, \star \tau)$ be named a topological approximation space induce by a finite group.

Proposition 4.2:

Theorem 4.2 (Conjugacy Topology Construction):

Let (G, *) be a finite group and let G / *R = { [x] : x in G } be the set of conjugacy classes of G. The collection τ_{conj} consisting of all arbitrary unions of conjugacy classes, together with the empty set, forms a valid topology on G.

Proof:

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To establish that τ_{conj} is a topology on G, we explicitly verify the topology axioms:

1. **Empty set and Universal set:** The empty set is in τ_{conj} as the empty union. Since conjugacy classes form a partition of G, the union of all conjugacy classes covers G. Thus G is in τ_{conj}.
2. **Arbitrary Unions:** Let {U_j} be an arbitrary family of open sets in τ_{conj}. Each U_j is a union of conjugacy classes. Therefore, the union of all U_j is also a union of conjugacy classes, which implies that it belongs to τ_{conj}.
3. **Finite Intersections:** Let U_1 and U_2 be open sets in τ_{conj}. Since conjugacy classes form an equivalence partition, any two classes are either identical or disjoint. Consequently, the intersection of U_1 and U_2 is either empty or a union of conjugacy classes, which means it belongs to τ_{conj}. By induction, finite intersections of open sets are open.

   Thus, τ_{conj}. is a valid topology on G.

Example using $\mathbb{Z} = 3\mathbb{Z}$:

- Group: G=($\{0,1,2\}, +_3$).
- Conjugacy Relation: In an Abelian Group, the conjugacy relation simplifies to $a \sim b$ iff $a = b$ because $a + g = g + b$ implies $a = b$. Thus, an equivalence classes are: $[0] = \{0\}, [1] = \{1\}, [2] = \{2\}$
- Base for Topology: The setoff equivalence classes $\{\{0\},\{1\},\{2\}\}$ forms a base for $\star\tau$.
- Topology $\star\tau$: All possible unions of base elements yield the discrete topology: $\star\tau = \{\emptyset,\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}\}$

Corollary 4.3:

Group $(.U,\star)$be Abel*i*an, at that time $\star R$ its' Identity Relation $([x] = \{x\} for\ all\ x)$, and $\star\tau$ is discrete topology (every subset is open).

Proof: In an Abelian group, $g \star x \star g^{-1} = x$ for all g, x. Thus, each element is only conjugacy to itself. The base consists of all singletons, generating the discrete topology.

Example using $\mathbb{Z} = 3\mathbb{Z}$:

- **Center of G:** Since $\mathbb{Z} = 3\mathbb{Z}$ is Abelian, $C(G) = G$
- **Topology** $\star\tau$: As shown above, $\star\tau$ is the discrete topology (every subset is open).

Corollary 4.4:

Every set in the base $\star B = U/\star R$ is clopen (both closed and open) in $\star\tau$.

Proof: Since $U/\star R$ is a partition the complement of any conjugacy class is the union of the other classes, which is open. Therefore, each conjugacy class is also closed.

Example using $\mathbb{Z} = 3\mathbb{Z}$:

- Subset $X = \{0,1\}$:
- Equivalence Classes: $\{0\}, \{1\}, \{2\}$
- Lower Approximation: $\underline{R}(X.) = .\{x \in U : [x] \subseteq X\} = \{0\} \cup \{1\} = \{0,1\}$
- Upper Approximation: $\overline{R}(X) = \{x \in U : [x] \cap X \neq \emptyset\} = \{0\} \cup \{1\} = \{0,1\}$
- Since, $\underline{R}(X) = \overline{R}(X), X$ is exact.
- Subset $Y = \{0\}$
- Lower Approximation: $\underline{R}(Y) = \{0\}$
- Upper Approximation: $\overline{R}(Y) = \{0\}$, Y is exact.

Theorem 4.5 (Equivalence of Rough Approximations and Topological Operators):

Let (G, *R, τ_{conj}) be a topological approximation space induced by a finite group G. For any subset X of G, the lower rough approximation and upper rough approximation coincide exactly with the topological interior and topological closure, respectively:

Lower Approximation of X = Int(X) and Upper Approximation of X = Cl(X)

Proof:

1. **Lower Approximation and Interior:** Int(X) is defined as the largest open set contained in X. Since every non-empty open set in τ_{conj} is a union of conjugacy classes, Int(X) is the largest union of conjugacy classes contained in X, which is exactly the definition of the lower rough approximation.
2. **Upper Approximation and Closure:** Cl(X) is defined as the smallest closed set containing X. In a partition topology, every open set is also closed. Thus, Cl(X) is the smallest union of conjugacy classes containing X, which strictly matches the definition of the upper rough approximation.

Definition 4.5 (Rough Set in $\star\tau$):

In the topological approximation space $(U,\star R,\star\tau)$, a set $X \subseteq U$ is a rough set if $\star\underline{R}(X) \neq \star\overline{R}(X)$. otherwise, it is an exact set.

Example 4.6:

Consider the quaternion group $Qs = \{\pm 1, \pm i, \pm j, \pm k\}$. Its conjugacy classes are: $\{-1\},\{1\},\{\pm i\},\{\pm j\},\{\pm K\}$. Thus the base for $\star\tau\ is\ \star B = \{\{1\},\{-1\},\{i,-i\},\{j,-j\},\{k,-k\}\}. Let\ X = \{i,j\}. Then$:

- $\star\underline{R}(X) = \emptyset$ (No full conjugacy class is contained in X).

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- $\star\overline{R}(X) = \{i, -i, j, -j\}$ (These classes have non-empty intersection with X).

  Since, $\star\underline{R.}(X) \neq .\star\overline{R}(X), X$ is rough set in $(Qs, \star R, \star\tau)$.

5. Rough Membership Function In $\star\tau$

We now define a rough membership function within the topological context.

Definition 5.1 (Topological Rough Membership Function):

Let $(U, \star R, \star\tau)$ be a topological approximation space and $X \subseteq U$. The rough membership function in $\star\tau$ is define as:

$$\mu_X^{\star\tau}: U \longrightarrow [0,1], \mu_X^{\star\tau}(x) = \frac{|B_x \cap X|}{|B_x|}$$

Where Bx is the smallest base element containing x. Since the base is a partition, $B_x = [x]_{\star_R}$ Therefore $\mu_X^{\star\tau}(x) = \mu_X^{\star R}(x)$

This definition seamlessly integrates yhe rough set and topological concepts.

Remark on Novelty and Topological Connection:

It is worth emphasizing the conceptual novelty of Definition 5.1 compared to Pawlak's classical rough membership function. In classical models, the membership degree relies solely on an external equivalence relation. In our topological framework, B_x = [x] is not merely an equivalence class, but represents the unique minimal open neighborhood containing x in τ_{conj}. The cardinality |B_x| is directly given by the index of the centralizer of element x in G. Thus, group symmetry directly dictates the granular resolution of membership probabilities.

Proposition 5.2:

Let $(U, \star R, \star\tau)$ be a topological approximation space and $X \subseteq U$. For the membership function …, the following hold:

i. If $\mu_X^{\star\tau}(x) = 1$, thus $x. \in \star\underline{R}(.X)$
ii. If $\mu_X^{\star\tau}(x) > 0, so\ x \in \star\overline{R}(X)$.
iii. If $\mu_X^{\star\tau}(x) = 0, whe\ x \notin \star\overline{R}(X)$.
iv. If $0 < \mu_X^{\star\tau}(x) < 1 for\ all\ x \in X, then\ \star\underline{R}(X) = \emptyset$

Example 5.3:

Let us return to Ds with conjugacy classes $\{e\}, \{r, r^2\}, \{s, sr, sr^2\}$. Let $X = \{r, s\}$.

We calculate the membership function:

i. $\mu_X^{\star\tau}(e) = \frac{|\{e\} \cap X|}{|\{e\}|} = \frac{0}{1} = 0$
ii. $\mu_X^{\star\tau}(r) = \frac{|\{r, r^2\} \cap X|}{|\{r, r^2\}|} = 0.5$
iii. $\mu_X^{\star\tau}(r^2) = 0.5$
iv. $\mu_X^{\star\tau}(s) = \frac{|\{s, rs, r^2s\} \cap X|}{|\{s, rs, r^2s\}|} = 0.333$
v. $\mu_X^{\star\tau}(rs) = 0.333$
vi. $\mu_X^{\star\tau}(r^2s) = 0.333$

Since, $0 < \mu_X^{\star\tau}(x) < 1$ for all $x \in X$, by proposition 5.2 (4), the lower approximation is empty: confirming X is a rough set.

Example 5.4 (Exact Set):

In the same space $(D3, \star R, \star\tau)$, let $Y = \{e, r, r^2\}$. Then:

i. $\mu_Y^{\star\tau}(e) = 1$
ii. $.\mu_Y^{\star\tau}(r) = 1$
iii. $.\mu_Y^{\star\tau}(r)^2 = 1$
iv. $\mu_Y^{\star\tau}(s) = 0$
v. $\mu_Y^{\star\tau}(rs) = 0$
vi. $\mu_Y^{\star\tau}(r^2s) = 0$

Now, $\star\underline{R}(Y) = \star\overline{R}(Y) = \{e, r, r^2\}$, so Y is an exact set.

6. Discussion

In this section, we provide a rigorous analysis of the algebraic-topological dynamics governing the proposed framework:

6.1 Algebraic Determinism of Granularity:

Information granularity is imposed externally through attribute relations in conventional rough set models. On the other hand, group algebra is the fundamental source of the granularity of the space (G, *R, τ_{conj}) The Class Equation states that each minimal open basis element B_x = [x] has a size equal to the index of its centralizer [G : C_G(x)]. This shows that elements with smaller centralizers produce coarser topological neighborhoods, while elements with larger centralizers provide finer neighborhoods..

6.2 Abelian vs. Non-Abelian Topology Dynamics:

A crucial insight of this framework is the distinct behavior between commutative and non-commutative groups:

1. **Abelian Groups:** If G is Abelian, the centralizer C_G(x) = G for all x in G, implying that every conjugacy class is a singleton (| [x] | = 1). The basis consists entirely of singletons, forcing τ_{conj} to be the discrete topology. In this case, the boundary region is empty for all subsets, eliminating rough uncertainty.
2. **Non-Abelian Groups:** For non-Abelian groups such as D_3 and Q_8, non-central elements produce conjugacy classes of cardinality greater than 1 (for example, | [r] | = 2 in D_3). This generates a non-discrete partition topology where non-trivial boundary regions arise naturally, providing a

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rich mathematical framework for modeling uncertainty.

6.3 Clopen Structural Stability:

Because the basis elements form a partition of G, every open set in τ_{conj} is simultaneously closed (clopen). This property guarantees that the boundary operator is always a union of whole conjugacy classes, ensuring topological structural stability without boundary fragmentation.

7. Conclusion

7.1 Summary of Established Mathematical Results:

In this research, we effectively constructed and formalized a topological approximation space (G, *R, τ_{conj}) based on finite group conjugacy relations. The key theoretical contributions presented in this paper encompass:

1. A rigorous demonstration that the conjugacy classes form a valid partition topology τ_{conj} on G (Theorem 4.2).
2. An established topological characterization theorem which shows that lower and upper rough approximations are formally equivalent to topological interior and closure operators (Theorem 4.5).
3. A proof indicating that the induced topology is discrete if and only if G is Abelian, while non-Abelian groups lead to non-discrete topologies with non-trivial rough boundary areas (Corollary 4.3 and Section 6).
4. The development of a topological rough membership function that is directly associated with centralizer indices (Definition 5.1).

7.2 Upcoming Projects and Possible Uses: Building on this fundamental framework, next studies will concentrate on:

1. Using Haar measures for continuous rough approximations, the framework is extended to infinite groups, namely Lie groups and compact topological groups.

2. Creating generalized algebraic rough sets using normalizers, automorphism groups, and subgroup conjugacy.

3. Examining applications in symmetric dataset classifications, cryptosystems, and algebraic coding theory.

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